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# Beilinson-Drinfeld chiral algebras for del Pezzo surfaces

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**Abstract**

Recently the chiral algebra of Beilinson-Drinfeld draws much attention in the mathematical physics of superstring theory. Naively, this is a holomorphic conformal field theory with integer graded conformal dimension, whose target space not necessarily has the vanishing first Chern class. This algebra has two ways of definition: one is that of Malikov-Schechtman-Vaintrob by gluing affine patches, and the other is that of Kapranov-Vasserot by gluing the formal loop space. We will use the method of Malikov-Schechtman-Vaintrob in order to compute the gerbes of chiral differential operators.

In this paper, we will examine the two independent ansatzes of Witten's (0,2) heterotic strings and Nekrasov's generalized complex geometry are consistent in the case of $\mathbb{CP}^2$, which has 3 affine patches and is expected to has the 1st Pontrjagin anomaly. We also extend this direction to the case of 2 dimensional toric Fano manifolds (toric del Pezzo surfaces) of all degrees, by blowing up the generic 1,2,3 points of $\mathbb{CP}^2$. These coincide with the computation of the Hirzebruch Riemann-Roch theorem. The most notable case is the 1 point blowup, where the total gauge invariant anomaly vanishes.

The significant future direction towards its application to the geometric Langlands program is also discussed in the last section.

# Contents

# Chapter 1

# Introduction

Discourse on Method (1637)
"Cogito ergo sum"
"Je pense, donc je suis"
*René* Descartes

Chiral de Rham complex of Beilinson and Drinfeld [BD] is an attempt to extend conformal field theory (CFT) to the case of target manifolds which do not necessarily obey the condition of the vanishing first Chern class of tangent bundle $c_1 = 0$. We usually exclude such a situation from our consideration since non-linear sigma model on such a target manifold contains a logarithmic divergence and hence a non-vanishing beta function. Scale invariance will be lost in such a situation and one can not apply the method of CFT.

One way of realizing the idea of Beilinson and Drinfeld is to use the patchwise construction by Malikov-Schectman-Vaintrob where one considers CFT defined on each coordinate patch and then considers the consistency of the theory under coordinate transformation among different patches. In the case when the manifold is covered by, say, 4 patches $U_0, U_1, U_2, U_3$, one first considers successive transformations $U_i \rightarrow U_{i+1}$. In the end one finds that under the total coordinate change $U_0 \rightarrow U_1 \rightarrow U_2 \rightarrow U_3 \rightarrow U_0$ the fields do not quite come back to their original values but there appears an additional term. Namely, there exists an obstruction or anomaly for a consistent CFT in such a system. It has been suggested that the obstruction is related to the first Potryagin class of the manifold [Gorbounov-Malikov-Schechtman, Witten, Nekrasov, et.al.].

In this paper we follow the approach of Malikov-Schechtman-Vaintrob[MSV] and study the case of del Pezzo surfaces which are rational surfaces obtained from $\mathbb{CP}^2$ by blowing up a certain number of points. We study the cases

of one-point, two-point, and three-point blowups of $\mathbb{CP}^2$ and show that the anomaly on these surfaces is in fact proportional to the first Pontrjagin class. We consider this is a substantial evidence in favor of the suggestion by Witten and Nekrasov. In chapter 2 of this these we start with the general theory of $\beta, \gamma$ system (conformal dimensions of $\gamma, \beta$ are 0 and 1, respectively) where $\gamma$ field is identified as the local coordinate of the manifold and $\beta$ field is identified an 1-form. Following [Malikov-Schechtman-Vaintrob] and Nekrasov we discuss the transformation laws of $\gamma, \beta$ system under coordinate change so that their OPE is preserved. Then in chapter 3 we discuss the case of $\mathbb{CP}^2$ as the target manifold and reproduce the result of Witten. In chapter 4 we consider the case of del Pezzo surface which is an 1,2,3-point blow up of $\mathbb{CP}^2$, which has the vanishing Pontrjagin anomaly in the 1 point case. We compute the obstruction for these cases and find that they are proportional to the first Pontrjagin class of the manifolds. Computations are somewhat lengthy but straightforward. Non-trivial aspect of the computation is the change in the convention of the normal ordering when one goes to a different coordinate patch and we have to make a careful analysis. In chapter 5 we present some discussions and conclusions. In Appendix A we explain the historical background of Wess-Zumino-Witten theory. This theory is used in the ansatz of Nekrasov(2.60) as the 1-form $B$. In Appendix B we include some brief illustrations of the toric diagrams and blowups.

## 1.1 The definition of del Pezzo surfaces

Del Pezzo surfaces are defined as the complex algebraic surfaces X such that the anticanonical divisor is ample. The anticanonical divisor of X is

$$\begin{aligned} -K_X &= 3H - E_1 - \cdots - E_n \\ H^2 &= 1 && (1.1) \\ E_i.E_j &= -\delta_{ij} && (1.2) \\ H.E_i &= 0, && (1.3) \end{aligned}$$

where $H$ is the hyperplane class and $E_i (i = 1, \cdots, n)$ is the exceptional divisor of $0 \leq n \leq 8$ generic point(s) blowups of $\mathbb{CP}^2$. Therefore we get $K_X^2 = 9 - n$. We will treat the generic $n$ ($n = 0, 1, 2, \cdots, 9$) points blowups of $\mathbb{CP}^2$ by attaching $\mathbb{CP}^1$ instead of points. It is differential geometrically, complex algebraic surfaces with positive curvatures.

# Chapter 2

# OPE of chiral de Rham complex: Malikov-Schechtman and Nekrasov

We will check the curved target space of $\beta\gamma$ CFT, which was examined by Witten[Witten] in the case of $\mathbb{CP}^2$.

## 2.1 Heisenberg algebra

[1]If we define the commutation relation,

$$\begin{aligned}
[\beta_{i\ n}, \gamma^j_m]_- &= \delta^j_i \delta_{m,-n} C && (2.1)\\
\gamma^i(z) &= \sum_{n\in\mathbb{Z}} \gamma^i_n z^{-n} && (2.2)\\
\beta_i(z) &= \sum_{n\in\mathbb{Z}} \beta_{i\ n} z^{-n-1}, && (2.3)
\end{aligned}$$

where $C$ is a constant that will be mapped to 1 later in the polynomial ring. $\gamma$ has conformal dimension 0, and $\beta$ has 1. Namely, $\gamma^i_n (n \geq 1)$ are the annihilation operators, and $\gamma^i_n (n \leq 0)$ are the creation operators. And $\beta_{i\ n} (n \geq 0)$ are the annihilation operators, and $\beta_{i\ n} (n \leq -1)$ are the creation operators. We will regard $\gamma^i$ as the coordinate, and $\beta_i$ are vector fields. For $x = \beta_{i\ n}$, or $\gamma^i_n (n \in \mathbb{Z})$ and $B \in End(V_N)$ ($V_N$ is the state space that the

[1]As compared to the original paper of Malikov-Schechtman-Vaintrob, change the notation such that $b \mapsto \gamma, a \mapsto \beta$ with a minus factor to the original paper

Heisenberg algebras act on), the normal ordered product $:xB:$ is giben by

$$
\begin{aligned}
:xB: \ &= \ Bx \ (if \ x \ is \ an \ annihilation \ operator) && (2.4)\\
& \quad\ xB \ otherwise. && (2.5)
\end{aligned}
$$

The OPE of $\beta$ and $\gamma$ is computed as follows

$$
\begin{aligned}
\beta_i(z)\gamma^j(w) &= \sum_{n,m} \beta_{i\ n}\gamma^j_m z^{-n-1}w^{-m} \\
&= \sum_{n<0,m} \beta_{i\ n}\gamma^j_m z^{-n-1}w^{-m} + \sum_{n\geq 0,m} \beta_{i\ n}\gamma^j_m z^{-n-1}w^{-m} \\
&= \sum_{n<0,m} \beta_{i\ n}\gamma^j_m z^{-n-1}w^{-m} + \sum_{n\geq 0,m} ([\beta_{i\ n},\gamma^j_m]_- z^{-n-1}w^{-m} + \gamma^j_m\beta_{i\ n} z^{-n-1}w^{-m}) \\
&= \sum_{n<0,m} \beta_{i\ n}\gamma^j_m z^{-n-1}w^{-m} + \sum_{n\geq 0,m} (\delta^j_i\delta_{n,-m} z^{-n-1}w^{-m} + \gamma^j_m\beta_{i\ n} z^{-n-1}w^{-m}) \\
&= \sum_{n<0,m} \beta_{i\ n}\gamma^j_m z^{-n-1}w^{-m} + \sum_{m\geq 0} \delta^j_i \frac{1}{z-w} + \sum_{n\geq 0,m} \gamma^j_m\beta_{i\ n} z^{-n-1}w^{-m} \\
&= \frac{\delta^j_i}{z-w} + :\beta_i(z)\gamma^j(w):. \qquad (2.6)
\end{aligned}
$$

Here it is assumed $|z| > |w|$ where the summation of the second term is absolutely convergent, and the first and the third term can be summarized as the normal ordering. We will call this the radial ordering in the following.

$$
\begin{aligned}
\beta_i(z)\gamma^j(w) \ &= \ \frac{\delta^j_i}{z-w} + (regular). && (2.7)\\
& && (2.8)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\beta_i(z)\beta_j(w) \ &= \ (regular) && (2.9)\\
\gamma^i(z)\gamma^j(w) \ &= \ (regular). && (2.10)
\end{aligned}
$$

The stress energy tensor is given by

$$
L(z) \ = \ :\partial_z\gamma(z)\beta(z):. \qquad (2.11)
$$

OPE of the stress-energy tensor is given by

$$
L(z)L(w) \ \sim \ \frac{1}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w}. \qquad (2.12)
$$

This can be verified by using the Wick theorem.

## 2.2 Clifford algebra

[2]Now we will consider the fermion fields $b^i, c_j$. The conformal dimension of $b^i$ and $c_j$ are 0 and 1, respectively. OPEs of $bc$-$\beta\gamma$ system are given by

$$\begin{aligned}
\beta_i(z)\gamma^j(w) &\sim \frac{\delta_i^j}{z-w} && (2.13)\\
\gamma^i(z)\gamma^j(w) &\sim 0 && (2.14)\\
\beta_i(z)\beta_j(w) &\sim 0 && (2.15)\\
b^i(z)c_j(w) &\sim \frac{\delta_j^i}{z-w} && (2.16)\\
b^i(z)b^j(w) &\sim 0 && (2.17)\\
c_i(z)c_j(w) &\sim 0 && (2.18)\\
\gamma^i(z)b^j(w) &\sim 0 && (2.19)\\
\gamma^i(z)c_j(w) &\sim 0 && (2.20)\\
\beta_i(z)b^j(w) &\sim 0 && (2.21)\\
\beta_i(z)c_j(w) &\sim 0. && (2.22)
\end{aligned}$$

## 2.3 A topological vertex algebra of rank N

In order to describe the topological vertex algebra, we introduce the following fields $L$: stress-energy tensor of conformal dimension 2, $J$: U(1) current of conformal dimension 1, and $Q, G$ are (twisted version of) two generators of $N = 2$ supersymmetry of conformal dimension 1 and 2, respectively. They are written as follows:

$$\begin{aligned}
L &= \sum_i \left[: \partial\gamma^i(z)\beta^i(z) : + : \partial b^i(z)c^i(z) :\right] && (2.23)\\
J &= \sum_i : b^i(z)c_i(z) : && (2.24)\\
Q &= \sum_i : \beta^i(z)b^i(z) : && (2.25)\\
G &= \sum_i : c_i(z)\partial\gamma^i(z) : . && (2.26)
\end{aligned}$$

The OPEs are given by

$$L(z)L(w) \sim \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w} \qquad (2.27)$$

[2]As compared to the original paper of Malikov-Schechtman-Vaintrob, we change the notation such that $\phi \mapsto b, \psi \mapsto c$.

$$J(z)J(w) \sim \frac{N}{(z-w)^2} \tag{2.28}$$

$$L(z)J(w) \sim -\frac{N}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w} \tag{2.29}$$

$$G(z)G(w) \sim 0 \tag{2.30}$$

$$L(z)G(w) \sim \frac{2G(w)}{(z-w)^2} + \frac{\partial_w G(w)}{z-w} \tag{2.31}$$

$$J(z)G(w) \sim -\frac{G(w)}{z-w} \tag{2.32}$$

$$Q(z)Q(w) \sim 0 \tag{2.33}$$

$$L(z)Q(w) \sim \frac{Q(w)}{(z-w)^2} + \frac{\partial_w Q(w)}{z-w} \tag{2.34}$$

$$J(z)Q(w) \sim \frac{Q(w)}{z-w} \tag{2.35}$$

$$Q(z)G(w) \sim \frac{N}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{L(w)}{z-w}. \tag{2.36}$$

Here $N$ denotes the dimension of the target manifolds. Note that the operator $L(z)$ can be expressed as the commutator. This is derived by integrating the equation (2.36) around w and we obtain

$$[Q_0, G(w)]_+ = L(w) \tag{2.37}$$

where $Q_0 = \oint Q(z)dz$ is the contour integration around the origin of $Q(z) = \sum_{n\in\mathbb{Z}} Q_n z^{n-1}$. Since the stress tensor $L(z)$ becomes the BRST exact, we obtain a topological theory. This algebra is determined by the twist of $N = 2$ SCFT, whose stress energy tensor is $T = L - \frac{1}{2}\partial J$.

## 2.4 Coordinate transformation

[3] The indices are important in this subsection. We will regard $\gamma^i$ as coordinates of the target space, $\beta_i$ and $c_i$ as 1-form, and $b^i$ as vector field. When we will change the coordinate $\gamma^i$ to $\tilde{\gamma^a}$, we have to determine the transformation of the other fields in such a way that they preserve the OPE. The result is given by

$$\tilde{\beta_i}(z) = :(\beta_j \frac{\partial \gamma^j}{\partial \tilde{\gamma^i}})(z): + :(\frac{\partial^2 \gamma^k}{\partial \tilde{\gamma^i} \partial \tilde{\gamma^l}} \frac{\partial \tilde{\gamma^l}}{\partial \gamma^r} b^r c_k)(z): \tag{2.38}$$

[3] As compared to the original paper of Malikov-Schechtman-Vaintrob, do the variable change of $f \mapsto \gamma, g \mapsto \tilde{\gamma}$.

$$\tilde{b}^i(z) = (\frac{\partial \tilde{\gamma}^i}{\partial \gamma^j} b^j)(z) \tag{2.39}$$

$$\tilde{c}_i(z) = (\frac{\partial \gamma^j}{\partial \tilde{\gamma}^i} c_j)(z). \tag{2.40}$$

Second term of $\tilde{\beta}^i$ (2.38) is the quantum effect. As we can easily verify it, the OPE of $bc$-$\beta\gamma$ system is preserved.

## 2.5 Useful OPEs

The purpose of this section is to derive useful formulae which will be used in the next section. In the next section we want to calculate the OPE for

$$\begin{aligned} \tilde{\beta}_a &:= \beta_i g_a^i + B_{ai}\partial\gamma^i \\ &= J_{g_a} + C_{B_a}, \end{aligned} \tag{2.41}$$

where the $g_a^i$ and $B_{ai}$ are the functions of $\gamma$. We define $J_V$ and $C_B$ for every 1-form $B \in \Omega_U^1$, and every vector $V \in \mathcal{T}_\mathcal{U}$, where $U$ is the coordinate patch which we are working with

$$\begin{aligned} J_V &= :\beta_i V^i(\gamma)(z): \\ &\equiv \lim_{\epsilon\to 0}[\beta_i(z+\epsilon)V^i(\gamma(z)) - \frac{1}{\epsilon}\partial_i V^i(\gamma(z))] \end{aligned} \tag{2.42}$$

$$C_B = B_i(\gamma(z))\partial\gamma^i. \tag{2.43}$$

Note that $C_B$ has the conformal weight 1, whereas $J_V$ has also conformal dimension 1 but has an extra $(z-w)^{-3}$ term in the OPE with the stress energy tensor $L(z)$.

By utilizing the inner product, the Lie derivatives, and the commutation relation in the basis of tangent bundle $\mathcal{T}_\mathcal{U}$, we obtain

$$[V_a, V_b]^j(z) = \partial_i V_b^j(z) V_a^i(z) - \partial_i V_a^j(z) V_b^i(z) \tag{2.44}$$

$$\mathcal{L}_V B(z) = \partial_i B_j(z) V^i(z) + \partial_j V^i(z) B_i(z). \tag{2.45}$$

We can compute the following OPEs

$$J_{V_a}(z+\epsilon)J_{V_b}(z) \sim -\frac{1}{2}\frac{\Sigma_{ab}(z+\epsilon)+\Sigma_{ab}(z)}{\epsilon^2} + \frac{J_{[V_a,V_b](z)}}{\epsilon} - \frac{C_{\Omega_{ab}}(z)}{\epsilon} \tag{2.46}$$

$$J_V(z+\epsilon)C_B(z) \sim \frac{\iota_V B(z)}{\epsilon^2} + \frac{C_{\mathcal{L}_V B(z)}}{\epsilon} \tag{2.47}$$

$$C_B(z+\epsilon)C_{B'}(z) \sim 0. \tag{2.48}$$

Here

$$\Sigma_{ab} = tr\mathcal{V}_a\mathcal{V}_b, \tag{2.49}$$

$$\Omega_{ab} = \frac{1}{2}tr(\mathcal{V}_a d\mathcal{V}_b - \mathcal{V}_b d\mathcal{V}_a), \tag{2.50}$$

where the matrix is defined by

$$(\mathcal{V}_a)_{ij} = \partial_i V_a^j. \tag{2.51}$$

We will use these equations in the following sections.

## 2.6 OPE on generalized complex manifolds

For later use, we will compute the OPE of combination of tangent and cotangent bundle. $V, W \in T_X$ and $\xi, \eta \in \Omega_X^1$

$$v = V \oplus \xi, \quad w = W \oplus \eta \in T_X \oplus \Omega_X^1 \tag{2.52}$$

$$\mathcal{O}_v := J_V + C_\xi \quad \mathcal{O}_w := J_W + C_\eta. \tag{2.53}$$

Using the formula of the last section

$$\begin{aligned}\mathcal{O}_v(z+\epsilon)\mathcal{O}_w(z) &= (J_V + C_\xi)(z+\epsilon)(J_W + C_\eta)(z) \\ &\sim J_V(z+\epsilon)J_W(z) + C_\xi(z+\epsilon)J_W(z) \\ &\quad + J_V(z+\epsilon)C_\eta(z).\end{aligned} \tag{2.54}$$

Then we use (2.46), (2.47) and obtain

$$\begin{aligned} LHS \sim & \left[-\frac{1}{2\epsilon^2}(\Sigma_{VW}(z+\epsilon) + \Sigma_{VW}(z)) + \frac{1}{\epsilon}J_{[V,W]}(z) - \frac{1}{\epsilon}C_{\Omega_{VW}}(z)\right] \\ + & \left[\frac{1}{(-\epsilon)^2}\iota_W\xi(z+\epsilon) + \frac{1}{(-\epsilon)}C_{\mathcal{L}_W\xi}(z+\epsilon)\right] \\ + & \left[\frac{1}{\epsilon^2}\iota_V\eta(z) + \frac{1}{\epsilon}C_{\mathcal{L}_V\eta}(z)\right]. \end{aligned} \tag{2.55}$$

By defining the metric in the generalized complex manifolds as follows

$$2g(v,w) := -\Sigma_{VW} + \iota_V\eta + \iota_W\xi \tag{2.56}$$

we can rewrite the $O(1/\epsilon^2)$ part of (2.55)

$$\frac{1}{\epsilon^2}(g(v,w)(z+\epsilon) + g(v,w)(z))$$

$$
\begin{aligned}
&= -\frac{1}{2\epsilon^2}\left[\Sigma_{VW}(z+\epsilon)+\Sigma_{VW}(z)\right] \\
&\quad +\frac{1}{2\epsilon^2}\left[\iota_V\eta(z+\epsilon)+\iota_V\eta(z)\right] \\
&\quad +\frac{1}{2\epsilon^2}\left[\iota_W\xi(z+\epsilon)+\iota_W\eta(z)\right]
\end{aligned}
\tag{2.57}
$$

(2.55) is then rewritten in the following way

$$
\begin{aligned}
LHS \quad \sim \quad & \frac{1}{\epsilon^2}(g(v,w)(z+\epsilon)+g(v,w)(z)) \\
- \quad & \frac{1}{2\epsilon^2}\left[\iota_V\eta(z+\epsilon)-\iota_V\eta(z)\right] \\
+ \quad & \frac{1}{2\epsilon^2}\left[\iota_W\xi(z+\epsilon)-\iota_W\eta(z)\right] \\
+ \quad & \frac{1}{\epsilon}J_{[V,W]}(z)-\frac{1}{\epsilon}C_{\Omega_{VW}}(z)-\frac{1}{\epsilon}C_{\mathcal{L}_W\xi}(z+\epsilon)+\frac{1}{\epsilon}C_{\mathcal{L}_V\eta}(z) \\
\sim \quad & \frac{1}{\epsilon^2}(g(v,w)(z+\epsilon)+g(v,w)(z)) \\
- \quad & \frac{1}{2\epsilon}C_{d\iota_V\eta(z)}+\frac{1}{2\epsilon}C_{d\iota_W\xi(z)} \\
+ \quad & \frac{1}{\epsilon}J_{[V,W]}(z)-\frac{1}{\epsilon}C_{\Omega_{VW}}(z)-\frac{1}{\epsilon}C_{\mathcal{L}_W\xi}(z)+\frac{1}{\epsilon}C_{\mathcal{L}_V\eta}(z) \\
= \quad & \frac{1}{\epsilon^2}(g(v,w)(z+\epsilon)+g(v,w)(z)) \\
+ \quad & \frac{1}{\epsilon}(\mathcal{O}_{[[v,w]]}-C_{\Omega_{VW}})
\end{aligned}
\tag{2.58}
$$

where we defined

$$
[[v,w]] \quad := \quad [V,W]+\mathcal{L}_V\eta-\mathcal{L}_W\xi-\frac{1}{2}d(\iota_V\eta-\iota_W\xi). \tag{2.59}
$$

## 2.7 The target space coordinate transformations

Let us introduce an ansatz of Nekrasov[N] for a coordinate change $\gamma^i$ to $\tilde{\gamma}^a$ such that the OPE of the $\beta\gamma$-system is unchanged after the coordinate transformation. We write the ansatz like as follows

$$
\begin{aligned}
\tilde{\beta}_a \quad &:= \quad \beta_i g_a^i + B_{ai}\partial\gamma^i \\
&= \quad J_{g_a}+C_{B_a}
\end{aligned}
\tag{2.60}
$$

where $B_a \in \Omega^1_U, g_a \in T_U$. Then by requiring

$$\tilde{\beta}_a(z)\tilde{\gamma}^b(w) \sim \frac{\delta^b_a}{z-w} \tag{2.61}$$

we obtain

$$g^i_a := \frac{\partial \gamma^i}{\partial \tilde{\gamma}^a}. \tag{2.62}$$

Now we examine the OPE relation

$$\tilde{\beta}_a \tilde{\beta}_b \sim 0. \tag{2.63}$$

By using the OPE of (2.58) of generalized complex manifolds of $v = (g_a \oplus B_a), w = (g_b \oplus B_b) \in T_U \oplus \Omega^1_U$ We obtain

$$\begin{aligned} \tilde{\beta}_a(z+\epsilon)\tilde{\beta}_b(z) &\sim \frac{1}{\epsilon^2}(g(v,w)(z+\epsilon) + g(v,w)(z)) \\ &+ \frac{1}{\epsilon}(\mathcal{O}_{[[v,w]]} - C_{\Omega_{ab}}). \end{aligned} \tag{2.64}$$

By requiring (2.63), the $O(1/\epsilon)$ and $O(1/\epsilon^2)$ terms have to vanish. As for $O(1/\epsilon^2)$ part, we have

$$\begin{aligned} g(v,w)(z) &= -\Sigma_{ab} + \iota_{g_a} B_b + \iota_{g_b} B_a \\ &= 0. \end{aligned} \tag{2.65}$$

We now want to determine $B_a$. Let us define the symmetric $\sigma_{ab}$ and anti-symmetric $\mu_{ab}$ part of $B_a$ as follows

$$B_a = \frac{1}{2}(\sigma_{ab} - \mu_{ab}) d\tilde{\gamma}^b. \tag{2.66}$$

Then, from the definition of symmetric part and (2.65), we can first conclude that

$$\begin{aligned} \sigma_{ab} = \sigma_{ba} &= \iota_{g_a} B_b + \iota_{g_b} B_a \\ &= \Sigma_{ab} = \sum_{i,j} \partial_i g^j_a \partial_j g^i_b \end{aligned} \tag{2.67}$$

We still have to determine $\mu_{ab}$. From the $O(1/\epsilon)$ part of $\tilde{\beta}\tilde{\beta}$ OPE (2.64), we obtain the necessary equation

$$\begin{aligned} \mathcal{O}_{[[v,w]]} - C_{\Omega_{ab}} &= J_{[g_a,g_b]} + C_{\mathcal{L}_{g_a} B_b - \mathcal{L}_{g_b} B_a - \frac{1}{2} d\mu_{ab}} - C_{\Omega_{ab}} \\ &= C_{\mathcal{L}_{g_a} B_b - \mathcal{L}_{g_b} B_a - \frac{1}{2} d\mu_{ab}} - C_{\Omega_{ab}} && (2.68) \\ &= 0. && (2.69) \end{aligned}$$

We contract this equation with any vector $g_c \in T_U$ to obtain the equation

$$\mathcal{L}_{g_c}\mu_{ab} + 2\iota_{g_c}\iota_{g_a}dB_b - 2\iota_{g_c}\iota_{g_b}dB_a \;\; = \;\; tr(\mathcal{G}_a\mathcal{L}_{g_c}\mathcal{G}_b - \mathcal{G}_b\mathcal{L}_{g_c}\mathcal{G}_a). \quad (2.70)$$

By using the Maurer-Cartan equation for $\mathcal{G}$, this equation can be rewritten as

$$\begin{aligned} d\mu &= -tr(d\tilde{g}^a{}_i(\tilde{g}^{-1})^i{}_b)^3 \\ &= tr(g^{-1}dg)^3, \end{aligned} \quad (2.71)$$

where $\tilde{g}^b{}_i g^i{}_a = \delta^b{}_a$ and there are no further conditions for $\mu$ besides this equation.

# Chapter 3

# Witten's $\mathbb{CP}^2$ case

## 3.1 Witten's $p_1$ anomaly for $\mathbb{CP}^2$

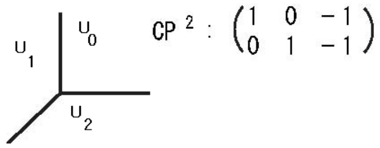


Figure 3.1: Toric diagram for $\mathbb{CP}^2$

Let's consider $\mathbb{CP}^2$ as the target space. We will review the computation of the $p_1$ anomaly of [Witten]. Let $U_\alpha \subset \mathbb{CP}^2$ be the affine coordinate patch defined by $\lambda^\alpha \neq 0$ $(\alpha = 0, 1, 2)$ for projective coordinate $(\lambda_0 : \lambda_1 : \lambda_2)$, Then, for the coordinate $\gamma^i$ $(i = 1, 2)$ of patch $U_\alpha$, we have the OPE of (2.13)

$$\beta_i(z)\gamma^j(z') \sim +\frac{\delta^i_j}{z - z'}. \tag{3.1}$$

Under the combined coordinate transformations from $U_0$ to $U_0$: $U_0 \to U_1 \to U_2 \to U_0$

$$\begin{aligned} \gamma^j &\to \gamma^j \\ \beta_i \to \beta'_i &= \beta_i + f_{ij}\partial\gamma^j \end{aligned} \tag{3.2}$$

where $f_{ij} = -f_{ji}$ is an antisymmetric tensor. Then,

$$\beta'_i(z')\beta'_j(z) \sim \frac{\partial\gamma^l}{z' - z}(\partial_i f_{jl} + \partial_j f_{li} + \partial_l f_{ij}) \tag{3.3}$$

We will denote $\gamma^1, \gamma^2$ as $v$, $w$ and $\beta_1, \beta_2$ as $V$, $W$. We assume that the generator of $H^2(\mathbb{CP}^2, \Omega^{2,cl})$ is given by

$$F = \frac{dv \wedge dw}{vw} \tag{3.4}$$

where $v$, $w$ are the inhomogeneous standard coordinate of $U_0 \subset \mathbb{CP}^2$.

$$v = \frac{\lambda_1}{\lambda_0} \tag{3.5}$$
$$w = \frac{\lambda_2}{\lambda_0}. \tag{3.6}$$

Then we perform successive coordinate transformations $U_0 \to U_1 \to U_2$ and back to $U_0$. We first check the coordinate transformation of $U_0 \to U_1$, which can be read from the toric diagram (Figure 3.1).

### 3.1.1 $U_0 \to U_1$

In the following subsections, as [Witten] did, we will use the notation $v^{[i]}, w^{[i]}, V^{[i]}, W^{[i]}$, instead of $\gamma, \beta$. To be more precise, let

$$v^{[i]} = \frac{\lambda_{i+1}}{\lambda_i} \tag{3.7}$$
$$w^{[i]} = \frac{\lambda_{i+2}}{\lambda_i} \tag{3.8}$$

$V^i, W^{[i]}$ are the corresponding 1-form.

$$v^{[1]} = \frac{w}{v} \tag{3.9}$$
$$w^{[1]} = \frac{1}{v} \tag{3.10}$$
$$V^{[1]} = vW \tag{3.11}$$
$$W^{[1]} = -v^2 V - vwW - \frac{5}{2}\partial v. \tag{3.12}$$

The Jacobian matrix $g_i^a$ is given by

$$\begin{aligned} g_i^a = (g)_{ia} &= \frac{\partial \tilde{\gamma}^a}{\partial \gamma^i} \\ &= \begin{pmatrix} -\frac{w}{v^2} & -\frac{1}{v^2} \\ \frac{1}{v} & 0 \end{pmatrix}_{ia}. \end{aligned} \tag{3.13}$$

Then the inverse matrix is¡

$$
\begin{aligned}
g_a^i = (g^{-1})_{ai} &= v^3 \begin{pmatrix} 0 & \frac{1}{v^2} \\ -\frac{1}{v} & -\frac{w}{v^2} \end{pmatrix}_{ai} \\
&= \begin{pmatrix} 0 & v \\ -v^2 & -vw \end{pmatrix}_{ai}
\end{aligned} \tag{3.14}
$$

This gives the Jacobian part $J_{g_a}$ of Nekrasov's formula (2.60) that assumes the generalized complex structure. Now that we have to determine the $B_{ai}$ of

$$
\begin{aligned}
\tilde{\beta}_a &:= \beta_i g_a^i + B_{ai}\partial\gamma^i \\
&= J_{g_a} + C_{B_a}.
\end{aligned} \tag{3.15}
$$

We decompose $B$ as $B_a = \frac{1}{2}(\sigma_{ab} - \mu_{ab})d\tilde{\gamma}^b$, where $d\mu = tr(g^{-1}dg)^3 = 0$ since $\mathbb{CP}^2$ is 2 dimensional. On the other hand, the symmetric part is given by

$$
\sigma_{ab} = \partial_i g_a^j \partial_j g_b^i. \tag{3.16}
$$

For $a = b = v^{[1]}$,

$$
\begin{aligned}
\sigma_{11} &= \partial_i g_1^j \partial_j g_1^i \\
&= \partial_v g_1^v \partial_v g_1^v \\
&= 0.
\end{aligned} \tag{3.17}
$$

For $a = v^{[1]}$ and $b = w^{[1]}$,

$$
\begin{aligned}
\sigma_{21} = \sigma_{12} &= \partial_i g_1^j \partial_j g_2^i \\
&= \partial_i g_1^w \partial_w g_2^i \\
&= \partial_v g_1^w \partial_w g_2^v \\
&= 0.
\end{aligned} \tag{3.18}
$$

Finally, $a = b = w^{[1]}$,

$$
\begin{aligned}
\sigma_{22} &= \partial_i g_2^j \partial_j g_2^i \\
&= 5v^2.
\end{aligned} \tag{3.19}
$$

Then,

$$
\begin{aligned}
(B)_{ia} = B_{ai} &= \frac{1}{2}\sigma_{ab}\frac{\partial\tilde{\gamma}^b}{\partial\gamma^i} \\
&= \frac{1}{2} g_i^b \sigma_{ab} \\
&= \frac{1}{2}\begin{pmatrix} -\frac{w}{v^2} & -\frac{1}{v^2} \\ \frac{1}{v} & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 5v^2 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & -5 \\ 0 & 0 \end{pmatrix}_{ia}.
\end{aligned} \tag{3.20}
$$

### 3.1.2 $\underline{U_0 \to U_1 \to U_2}$

Similar discussions yield that the coordinate transformations for $U_1 \to U_2$

$$
\begin{aligned}
v^{[2]} &= \frac{w^{[1]}}{v^{[1]}} = \frac{1}{w} && (3.21)\\
w^{[2]} &= \frac{1}{v^{[1]}} = \frac{v}{w} && (3.22)\\
V^{[2]} &= v^{[1]}W^{[1]} && (3.23)\\
W^{[2]} &= -(v^{[1]})^2V^{[1]} - v^{[1]}w^{[1]}W^{[1]} - \frac{5}{2}\partial v^{[1]}, && (3.24)
\end{aligned}
$$

and $U_2 \to U_0$

$$
\begin{aligned}
v^{[3]} &= \frac{w^{[2]}}{v^{[2]}} = v && (3.25)\\
w^{[3]} &= \frac{1}{v^{[2]}} = w && (3.26)\\
V^{[3]} &= v^{[2]}W^{[2]} && (3.27)\\
W^{[3]} &= -(v^{[2]})^2V^{[2]} - v^{[2]}w^{[2]}W^{[2]} - \frac{5}{2}\partial v^{[2]} && (3.28)
\end{aligned}
$$

To be more careful, when we substitute and combine the equations (3.11)(3.12) ($U_0 \to U_1$)in the equations of $V^{[2]}$ (3.23) and $W^{[2]}$ (3.24) for $U_1 \to U_2$, which have the cross terms with $\frac{1}{v}$ or $\frac{w}{v}$. Since the definition of normal product depends on the patches and each term of (3.24) is defined by normal product, the first term of $W^{[2]}$, for example, is defined as

$$
\begin{aligned}
-(v^{[1]})^2V^{[1]} &= \lim_{z'\to z}(-v^{[1]}(z')^2V^{[1]}(z) + 2v^{[1]}(z')\frac{1}{z'-z})\\
&= \lim_{z'\to z}\left[-(\frac{w(z')}{v(z')})^2v(z)W(z) + 2(\frac{w(z')}{v(z')})\frac{-1}{z'-z})\right]\\
&= \lim_{z'\to z}\left[-\frac{2w(z')}{z-z'}\frac{v(z)}{v(z')^2} - (\frac{w(z')}{v(z')})^2v(z)W(z) + 2(\frac{w(z')}{v(z')})\frac{-1}{z'-z}\right]\\
&= \lim_{\epsilon\to 0}\left[(+\frac{2}{\epsilon}\frac{w}{v} - 4\frac{w\partial v}{v^2} + \frac{2\partial w}{v}) - \frac{w^2}{v}W + (\frac{-2(\partial wv - \partial vw)}{v^2} - \frac{2}{\epsilon}\frac{w}{v}))\right]\\
&= -2\frac{\partial vw}{v^2} - \frac{w^2}{v}W && (3.29)
\end{aligned}
$$

By applying similar treatments for the 2nd term of (3.24), we obtained the relation for $U_0 \to U_1 \to U_2$. Therefore

$$
\begin{aligned}
V^{[2]} &= -vwV - w^2W - \frac{3w\partial v}{2v} - \partial w && (3.30)\\
W^{[2]} &= wV - \frac{3\partial w}{2v} && (3.31)
\end{aligned}
$$

### 3.1.3 $\underline{U_0 \to U_1 \to U_2 \to U_0}$

Again, we will substitute the $V^{[3]}, W^{[3]}$ by the equations above to get

$$\begin{aligned} V^{[3]} &= v^{[2]} W^{[2]} \\ &= \frac{1}{w} : wV - \frac{3}{2}\frac{\partial w}{v} : \\ &= V - \frac{3}{2}\frac{\partial w}{vw}, \end{aligned} \tag{3.32}$$

and for $z' = z + \epsilon$

$$\begin{aligned} W^{[3]} &= - : (v^{[2]})^2 V^{[2]} : - : v^{[2]} w^{[2]} W^{[2]} : - \frac{5}{2}\partial v^{[2]} \\ &= - \lim_{\epsilon\to 0}\left[(v^{[2]}(z'))^2 V^{[2]}(z) + 2V^{[2]}(z)\frac{1}{\epsilon} + 2\partial v^{[2]}\right] \\ &\quad - \lim_{\epsilon\to 0}\left[v^{[2]}(z') w^{[2]}(z') W^{[2]}(z) + v^{[2]}\frac{1}{\epsilon} + \partial v^{[2]}\right] \\ &\quad - \frac{5}{2}\partial v^{[2]} \\ &= \lim_{\epsilon\to 0}\left[-\frac{3}{\epsilon} v^{[2]}(z) - \frac{11}{2}\partial v^{[2]} - (v^{[2]}(z'))^2 V^{[2]}(z) - v^{[2]}(z') w^{[2]}(z') W^{[2]}(z)\right] \\ &= \lim_{\epsilon\to 0}\left[-\frac{5}{2}\partial v^{[2]} + \frac{1}{w(z')^2}(vwV + w^2 W + \frac{3w\partial v}{2v} + \partial w)(z)\right] \\ &\quad + \lim_{\epsilon\to 0}\left[-\frac{1}{w(z')}\frac{v(z')}{w(z')}(wV - \frac{3}{2}\frac{\partial w}{v})(z)\right] \\ &= \lim_{\epsilon\to 0}\left[-\frac{3}{\epsilon} v^{[2]}(z) - \frac{11}{2}\partial v^{[2]} + \frac{vV}{w} + \frac{2w^{-3}(z') w^2(z)}{\epsilon} + \frac{3\partial v}{2vw}\right] \\ &\quad + \lim_{\epsilon\to 0}\left[\frac{\partial w}{w^2} + \frac{w(z)}{w(z')^2}\frac{1}{\epsilon} - \frac{vV}{w} + \frac{3\partial w}{2w^2} + W(z)\right] \\ &= W(z) + \frac{3}{2}\frac{\partial v}{vw} \end{aligned} \tag{3.33}$$

Thereby, in total,

$$\begin{aligned} v &\to v \\ w &\to w \\ V &\to V - \frac{3}{2}\frac{\partial w}{vw} \\ W &\to W + \frac{3}{2}\frac{\partial v}{vw} \end{aligned} \tag{3.34}$$

This certainly is the ansatz of Witten (3.2). We will check whether this result can be derived from the ansatz of Nekrasov (2.60) with the $d\mu = 0$ whereas the total gerbe term $f_{ij}$ is antisymmetric. This is due to the anomaly 2-form of Nekrasov.

## 3.2 Anomaly 2-form $\psi_{\alpha\beta\gamma}$

For $U_\alpha, U_\beta, U_\gamma$: affine patches, we perform successive coordinate transformations $U_\alpha \to U_\beta \to U_\gamma \to U_\alpha$. The coordinate transformations in each step are given by

$$\begin{aligned} U_\alpha \to U_\beta \quad &: \quad \gamma^i \to \gamma^a, \quad \beta^i \to \beta^a = g_i^a \beta^i + B_{ai}\gamma^i, & (3.35)\\ U_\beta \to U_\gamma \quad &: \quad \gamma^a \to \gamma^\alpha, \quad \beta^a \to \beta^\alpha = g_a^\alpha \beta^a + B'_{\alpha a}\gamma^a, & (3.36)\\ U_\gamma \to U_\alpha \quad &: \quad \gamma^\alpha \to \gamma^I, \quad \beta^\alpha \to \beta^I = g_\alpha^I \beta^\alpha + B''_{I\alpha}\gamma^\alpha. & (3.37)\end{aligned}$$

We will use the indices $i, j, k, ..., a, b, c, ..., \alpha, \beta, \gamma, ..., I, J, K, ...$ as

$$\gamma^i, \gamma^j, \gamma^k \in U_\alpha \tag{3.38}$$
$$\gamma^a, \gamma^b, \gamma^c \in U_\beta \tag{3.39}$$
$$\gamma^\alpha, \gamma^\beta, \gamma^\gamma \in U_\gamma \tag{3.40}$$
$$\gamma^I, \gamma^J, \gamma^K \in U_\alpha. \tag{3.41}$$

By combining these three coordinate transformations, we obtain

$$\gamma^i \quad \to \quad \gamma^i \tag{3.42}$$
$$\beta_j \quad \to \quad \beta_j - \frac{1}{2}(\psi_{\alpha\beta\gamma})_{Ij}\partial\gamma^I. \tag{3.43}$$

$\psi_{\alpha\beta\gamma}$ is defined by

$$\begin{aligned} \psi_{\alpha\beta\gamma} \quad &= \quad \mu_{\alpha\beta} + \mu_{\beta\gamma} + \mu_{\gamma\alpha} - tr(g''dg' \wedge dg) \\ &=: \quad \mu_{\alpha\beta} + \mu_{\beta\gamma} + \mu_{\gamma\alpha} + \psi^0_{\alpha\beta\gamma} \\ &\qquad \in H^2(X, T_X \oplus \Omega^2_X/d\Omega^1_X), & (3.44)\end{aligned}$$

where $\mu_{\alpha\beta}$, $\mu_{\beta\gamma}$, and $\mu_{\gamma\alpha}$ are the antisymmetric parts of $B$, $B'$, and $B''$. This is called the anomaly 2-form. (3.43) can be proved as follows; for 2 step coordinate changes $U_\alpha \to U_\beta \to U_\gamma$,

$$\begin{aligned} \beta'_\alpha \quad &= \quad g_\alpha^a \tilde{\beta}_a + B'_{\alpha a}\partial\gamma^a \\ &= \quad \lim_{\epsilon\to 0}\left[g_\alpha^a(z+\epsilon)\tilde{\beta}_a(z) + \frac{1}{\epsilon}\partial_a g_\alpha^a(z+\epsilon)\right]\end{aligned}$$

$$\begin{aligned}
&= \lim_{\epsilon\to 0}\left[g^a_\alpha(z+\epsilon)(g^i_a(z)\beta_i(z)+B_{ai}(z)\partial\gamma^i(z))+\frac{1}{\epsilon}\partial_a g^a_\alpha(z+\epsilon)\right] \\
&= \lim_{\epsilon\to 0}\left[g^a_\alpha g^i_a\beta_i+g^a_\alpha B_{ai}(z)\partial\gamma^i(z)-\frac{1}{\epsilon}\partial_i g^a_\alpha(z+\epsilon)g^i_a(z)+\frac{1}{\epsilon}\partial_a g^a_\alpha(z+\epsilon)\right].
\end{aligned} \tag{3.45}$$

This can be rewritten as follows by using $g^a_\alpha g^i_a = g^i_\alpha$, $g^i_a(z)\sim g^i_a(z+\epsilon)-\partial_z g^i_a\epsilon$ and $\partial_i g^a_\alpha(z+\epsilon)g^i_a(z+\epsilon)=\partial_a g^a_\alpha(z+\epsilon)$,

$$\begin{aligned}
\beta'_\alpha &= g^i_\alpha\beta_i+g^a_\alpha B_{ai}\partial\gamma^i+B'_{\alpha a}g^a_i\partial\gamma^i+\partial_i g^a_\alpha\partial_j g^i_\alpha\partial\gamma^j \\
&= g^i_\alpha\beta_i+\hat{B}_{\alpha i}\partial\gamma^i,
\end{aligned} \tag{3.46}$$

where $\hat{B}_{\alpha i}$ is given by

$$\hat{B}_{\alpha i} \;:=\; (B_{\alpha i}+B'_{\alpha i}+\partial_j g^a_\alpha\partial_i g^j_a)\partial\gamma^i. \tag{3.47}$$

The 3 step coordinate change $U_\alpha\to U_\beta\to U_\gamma\to U_\alpha$ (3.43) is obtained by combining (3.46) and the (3.37).

Let us confirm this reproduce the previous result for the case of $\mathbb{CP}^2$. $d\mu=0$ and we can set $\mu$ as arbitrary value. We have to check whether $\psi_{\alpha\beta\gamma}=0$ for some $\mu$ after computing $\psi^0_{\alpha\beta\gamma}$

$$\begin{aligned}
(g^i_a)_{ai} &= \begin{pmatrix} 0 & v \\ -v^2 & -vw \end{pmatrix} \\
(g'^a_\alpha)_{\alpha a} &= \begin{pmatrix} 0 & v^{[1]} \\ -(v^{[1]})^2 & -v^{[1]}w^{[1]} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{w}{v} \\ -\frac{w^2}{v^2} & -\frac{w}{v^2} \end{pmatrix} \\
(g''^\alpha_i)_{i\alpha} &= \begin{pmatrix} 0 & v^{[2]} \\ -(v^{[2]})^2 & -v^{[2]}w^{[2]} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{1}{w} \\ -\frac{1}{w^2} & -\frac{v}{w^2} \end{pmatrix}
\end{aligned} \tag{3.48}$$

Therefore

$$\begin{aligned}
dg &= dv\begin{pmatrix} 0 & 1 \\ -2v & -w \end{pmatrix}+dw\begin{pmatrix} 0 & 0 \\ 0 & -v \end{pmatrix} \\
dg' &= dv\begin{pmatrix} 0 & -\frac{w}{v^2} \\ 2\frac{w^2}{v^3} & 2\frac{w}{v^3} \end{pmatrix}+dw\begin{pmatrix} 0 & \frac{1}{v} \\ -\frac{2w}{v^2} & -\frac{1}{v^2} \end{pmatrix}
\end{aligned} \tag{3.49}$$

and

$$
\begin{aligned}
dg' \wedge dg &= dv \wedge dw \begin{pmatrix} 2 & 2w/v \\ 2/v & -w/v^2 \end{pmatrix} \\
trg'' dg' \wedge dg &= dv \wedge dw (\frac{2}{vw} + \frac{1}{vw}) = \frac{3 dv \wedge dw}{vw}
\end{aligned} \tag{3.50}
$$

This agrees with the result of direct OPE computation in (3.34). And the anomaly exists because $\mu$ cannot eliminate this term.

# Chapter 4

# Case of del Pezzo surfaces

## 4.1 Anomaly 2-forms for 1 point blowup of $\mathbb{CP}^2$

The coordinate change can be read from the toric diagram (Figure 4.1) and (Figure 4.2)

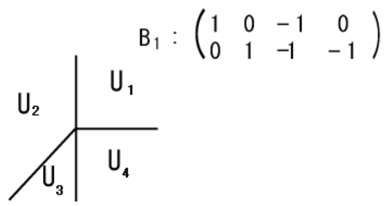


Figure 4.1: Toric diagram for 1 point blowup of $\mathbb{CP}^2$

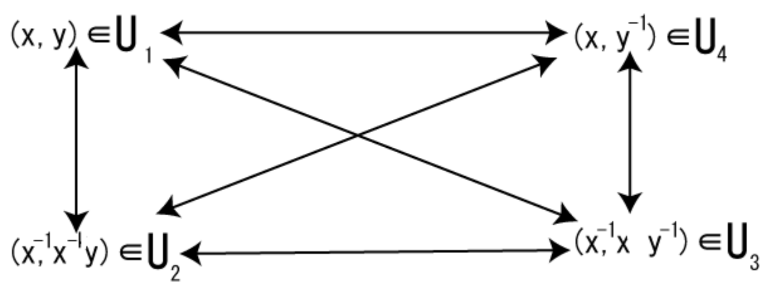


Figure 4.2: Coordinate changes for the 1 point blowup of $\mathbb{CP}^2$

$$\begin{aligned}
\gamma_1^{[1]} &= x = (\gamma_1^{[3]})^{-1} \\
\gamma_2^{[1]} &= y = (\gamma_1^{[3]})^{-1}(\gamma_2^{[3]})^{-1} \\
\gamma_1^{[2]} &= x^{-1} \\
\gamma_2^{[2]} &= x^{-1}y \\
\gamma_1^{[3]} &= x^{-1} = \gamma_1^{[2]} \\
\gamma_2^{[3]} &= xy^{-1} = (\gamma_2^{[2]})^{-1} \qquad (4.1)
\end{aligned}$$

The Jacobian matrices (note that we are not dealing with inverse Jacobians) for $U_1 \to U_2 \to U_3 \to U_1$ are given by

$$\begin{aligned}
(g_i^a)_{ia} &= \frac{\partial \gamma_a^{[2]}}{\partial \gamma_i^{[1]}} \\
&= \begin{pmatrix} -x^{-2} & -x^{-2}y \\ 0 & x^{-1} \end{pmatrix} \\
(g_a'^{\alpha})_{a\alpha} &= \frac{\partial \gamma_\alpha^{[3]}}{\partial \gamma_a^{[2]}} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -x^2y^{-2} \end{pmatrix} \\
(g_\alpha''^{i})_{\alpha i} &= \frac{\partial \gamma_i^{[1]}}{\partial \gamma_\alpha^{[3]}} \\
&= \begin{pmatrix} -x^2 & -xy \\ 0 & -x^{-1}y^2 \end{pmatrix} \qquad (4.2)
\end{aligned}$$

Then

$$\begin{aligned}
dg &= dx \begin{pmatrix} 2x^{-3} & 2x^{-3}y \\ 0 & -x^{-2} \end{pmatrix} + dy \begin{pmatrix} 0 & -x^{-2} \\ 0 & 0 \end{pmatrix} \\
dg' &= dx \begin{pmatrix} 0 & 0 \\ 0 & -2xy^{-2} \end{pmatrix} + dy \begin{pmatrix} 0 & 0 \\ 0 & 2x^2y^{-3} \end{pmatrix} \qquad (4.3)
\end{aligned}$$

Therefore

$$\begin{aligned}
dg \wedge dg' &= dx \wedge dy \left[ \begin{pmatrix} 0 & 4x^{-1}y^{-2} \\ 0 & -2y^{-3} \end{pmatrix} - \begin{pmatrix} 0 & 2x^{-1}y^{-2} \\ 0 & 0 \end{pmatrix} \right] \\
&= dx \wedge dy \begin{pmatrix} 0 & 2x^{-1}y^{-2} \\ 0 & -2y^{-3} \end{pmatrix} \\
\psi_{123}^0 = tr g'' dg \wedge dg' &= 2\frac{dx \wedge dy}{xy} \qquad (4.4)
\end{aligned}$$

Similarly, for $U_1 \to U_2 \to U_4 \to U_1$

$$\begin{aligned}
\gamma_1^{[1]} &= x = \gamma_1^{[4]} && (4.5)\\
\gamma_2^{[1]} &= y = (\gamma_2^{[4]})^{-1} && (4.6)\\
\gamma_1^{[2]} &= x^{-1} && (4.7)\\
\gamma_2^{[2]} &= x^{-1}y && (4.8)\\
\gamma_1^{[4]} &= x = (\gamma_1^{[2]})^{-1} && (4.9)\\
\gamma_2^{[4]} &= y^{-1} = \gamma_1^{[2]}(\gamma_2^{[2]})^{-1} && (4.10)
\end{aligned}$$

and the Jacobians are

$$\begin{aligned}
(h_i^a) &= \frac{\partial\gamma_a^{[2]}}{\partial\gamma_i^{[1]}}\\
&= \begin{pmatrix} -x^{-2} & -x^{-2}y \\ 0 & x^{-1} \end{pmatrix}\\
(h_a'^{\alpha}) &= \frac{\partial\gamma_\alpha^{[4]}}{\partial\gamma_a^{[2]}}\\
&= \begin{pmatrix} -(\gamma_1^{[2]})^{-2} & (\gamma_2^{[2]})^{-1} \\ 0 & -\gamma_1^{[2]}(\gamma_2^{[2]})^{-1} \end{pmatrix}\\
&= \begin{pmatrix} -x^2 & xy^{-1} \\ 0 & -xy^{-2} \end{pmatrix}\\
(h_\alpha''^{i}) &= \frac{\partial\gamma_i^{[1]}}{\partial\gamma_\alpha^{[4]}}\\
&= \begin{pmatrix} 1 & 0 \\ 0 & -(\gamma^{[4]})^{-2} \end{pmatrix}\\
&= \begin{pmatrix} 1 & 0 \\ 0 & -y^2 \end{pmatrix} && (4.11)
\end{aligned}$$

Then

$$\begin{aligned}
dh &= dx\begin{pmatrix} 2x^{-3} & 2x^{-3}y \\ 0 & -x^{-2} \end{pmatrix} + dy\begin{pmatrix} 0 & x^{-2} \\ 0 & 0 \end{pmatrix}\\
dh' &= dx\begin{pmatrix} -2x & y^{-1} \\ 0 & -y^{-2} \end{pmatrix} + dy\begin{pmatrix} 0 & -xy^{-2} \\ 0 & 2xy^{-3} \end{pmatrix} && (4.12)
\end{aligned}$$

Therefore

$$dh \wedge dh' = dx \wedge dy\left[\begin{pmatrix} 0 & 2x^{-2}y^{-2} \\ 0 & 2x^{-1}y^{-3} \end{pmatrix} - \begin{pmatrix} 0 & x^{-2}y^{-2} \\ 0 & 0 \end{pmatrix}\right]$$

$$
\begin{aligned}
&= dx \wedge dy \begin{pmatrix} 0 & x^{-2}y^{-2} \\ 0 & 2x^{-1}y^{-3} \end{pmatrix} \\
\psi^0_{124} = trh''dh \wedge dh' &= 2\frac{dx \wedge dy}{xy}. \qquad (4.13)
\end{aligned}
$$

For $U_1 \longrightarrow U_3 \longrightarrow U_4 \longrightarrow U_1$

$$
\begin{aligned}
\gamma_1^{[1]} &= x = \gamma_1^{[4]} && (4.14) \\
\gamma_2^{[1]} &= y = (\gamma_2^{[4]})^{-1} && (4.15) \\
\gamma_1^{[3]} &= x^{-1} && (4.16) \\
\gamma_2^{[3]} &= xy^{-1} && (4.17) \\
\gamma_1^{[4]} &= x = (\gamma_1^{[3]})^{-1} && (4.18) \\
\gamma_2^{[4]} &= y^{-1} = \gamma_1^{[3]}\gamma_2^{[3]} && (4.19)
\end{aligned}
$$

and the Jacobians are

$$
\begin{aligned}
(k_i^a) &= \frac{\partial \gamma_a^{[3]}}{\partial \gamma_i^{[1]}} \\
&= \begin{pmatrix} -x^{-2} & -y^{-1} \\ 0 & xy^{-2} \end{pmatrix} \\
(k_a'^{\alpha}) &= \frac{\partial \gamma_\alpha^{[4]}}{\partial \gamma_a^{[3]}} \\
&= \begin{pmatrix} -(\gamma_1^{[3]})^{-2} & \gamma_2^{[3]} \\ 0 & \gamma_2^{[3]} \end{pmatrix} \\
&= \begin{pmatrix} -x^2 & xy^{-1} \\ 0 & -x^{-1} \end{pmatrix} \\
(k_\alpha''^{i}) &= \frac{\partial \gamma_i^{[1]}}{\partial \gamma_\alpha^{[4]}} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -(\gamma_2^{[4]})^{-2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -y^2 \end{pmatrix} \qquad (4.20)
\end{aligned}
$$

Then

$$
\begin{aligned}
dk &= dx \begin{pmatrix} 2x^{-3} & 0 \\ 0 & -y^{-2} \end{pmatrix} + dy \begin{pmatrix} 0 & y^{-2} \\ 0 & 2xy^{-3} \end{pmatrix} \\
dk' &= dx \begin{pmatrix} -2x & y^{-1} \\ 0 & -x^{-2} \end{pmatrix} + dy \begin{pmatrix} 0 & -xy^{-2} \\ 0 & 0 \end{pmatrix} \qquad (4.21)
\end{aligned}
$$

Therefore

$$\begin{aligned}
dk \wedge dk' &= dx \wedge dy \left[ \begin{pmatrix} 0 & -2x^{-2}y^{-2} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & x^{-2}y^{-2} \\ 0 & -2x^{-1}y^{-3} \end{pmatrix} \right] \\
&= dx \wedge dy \begin{pmatrix} 0 & -3x^{-2}y^{-2} \\ 0 & 2x^{-1}y^{-3} \end{pmatrix} \\
\psi^0_{134} = trk''dk \wedge dk' &= -2\frac{dx \wedge dy}{xy}. \qquad (4.22)
\end{aligned}$$

For $U_2 \longrightarrow U_3 \longrightarrow U_4 \longrightarrow U_2$

$$\begin{aligned}
\gamma_1^{[2]} &= x^{-1} = (\gamma_1^{[4]})^{-1} && (4.23) \\
\gamma_2^{[2]} &= x^{-1}y = (\gamma_1^{[4]})^{-1}(\gamma_2^{[4]})^{-1} && (4.24) \\
\gamma_1^{[3]} &= x^{-1} = \gamma_1^{[2]} && (4.25) \\
\gamma_2^{[3]} &= xy^{-1} = (\gamma_2^{[2]})^{-1} && (4.26) \\
\gamma_1^{[4]} &= x = (\gamma_1^{[3]})^{-1} && (4.27) \\
\gamma_2^{[4]} &= y^{-1} = \gamma_1^{[3]}\gamma_2^{[3]} && (4.28)
\end{aligned}$$

and the Jacobians are

$$\begin{aligned}
(l_i^a) &= \frac{\partial\gamma_a^{[3]}}{\partial\gamma_i^{[2]}} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -(\gamma_2^{[2]})^{-2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -x^2y^{-2} \end{pmatrix} \\
(l'^{\alpha}_a) &= \frac{\partial\gamma_\alpha^{[4]}}{\partial\gamma_a^{[3]}} \\
&= \begin{pmatrix} -(\gamma_1^{[3]})^{-2} & \gamma_2^{[3]} \\ 0 & \gamma_2^{[3]} \end{pmatrix} \\
&= \begin{pmatrix} -x^2 & xy^{-1} \\ 0 & -x^{-1} \end{pmatrix} \\
(l''^i_\alpha) &= \frac{\partial\gamma_i^{[2]}}{\partial\gamma_\alpha^{[4]}} \\
&= \begin{pmatrix} -(\gamma_1^{[4]})^{-2} & -(\gamma_1^{[4]})^{-2}(\gamma_2^{[4]})^{-1} \\ 0 & -(\gamma_1^{[4]})^{-1}(\gamma_2^{[4]})^{-2} \end{pmatrix} \\
&= \begin{pmatrix} -x^{-2} & -x^{-2}y \\ 0 & -x^{-2}y^2 \end{pmatrix} \qquad (4.29)
\end{aligned}$$

Then

$$
\begin{aligned}
dl &= dx \begin{pmatrix} 0 & 0 \\ 0 & -2xy^{-2} \end{pmatrix} + dy \begin{pmatrix} 0 & 0 \\ 0 & 2x^2y^{-3} \end{pmatrix} \\
dl' &= dx \begin{pmatrix} -2x & y^{-1} \\ 0 & -x^{-2} \end{pmatrix} + dy \begin{pmatrix} 0 & -xy^{-2} \\ 0 & 0 \end{pmatrix}
\end{aligned} \tag{4.30}
$$

Therefore

$$
dl \wedge dl' = dx \wedge dy \left[ 0 - \begin{pmatrix} 0 & 0 \\ 0 & -2y^{-3} \end{pmatrix} \right] \tag{4.31}
$$

$$
= dx \wedge dy \begin{pmatrix} 0 & 0 \\ 0 & 2y^{-3} \end{pmatrix}
$$

$$
\psi^0_{234} = trl'' dl \wedge dl' = -2\frac{dx \wedge dy}{xy} \tag{4.32}
$$

## 4.2 Antisymmetric $\mu$-term for 1 point blowup of $\mathbb{CP}^2$

In the previous section, we obtained the last term $\psi^0_{\alpha\beta\gamma}$ in the anomaly 2-form (3.44) as

$$
\psi^0_{\alpha\beta\gamma} = c_{\alpha\beta\gamma}\frac{dx \wedge dy}{xy}|_{U_\alpha \cap U_\beta \cap U_\gamma}, \tag{4.33}
$$

where the coefficients $c_{\alpha\beta\gamma}$ are given by

$$
c_{123} = 2, \quad c_{124} = 2, \quad c_{134} = -2, \quad c_{234} = -2. \tag{4.34}
$$

In this section we discuss the other three $\mu$-terms in (3.44). If $\psi^0_{\alpha\beta\gamma}$ is cancelled by choosing appropriate $\mu$-terms, the anomaly is absent. This is indeed the case for the 1 point blowup of $\mathbb{CP}^2$.

This is well described in terms of Cech cohomology.

$$
H^2(X, \Omega^{cl}) = \frac{Ker(\delta : C^2 \to C^3)}{Im(\delta : C^1 \to C^2)}, \tag{4.35}
$$

where X is the 1 point blowup of $\mathbb{CP}^2$ and $\Omega^{cl}$ is the chiral de Rham complex and $C^i$ is $\Omega^{cl}$'s i-th Cech complex and $\delta$ is the corresponding coboundary operator. $\psi^0_{\alpha\beta\gamma}$ is described as the 2-cochain $c^2$.

$$
(c^2)_{U_i \cap U_j \cap U_k} = c_{ijk} \tag{4.36}
$$

The $\mu$-terms are 1-cochain. Here we only consider $\mu_{\alpha\beta}$ in the form of

$$\mu_{\alpha\beta} = c_{\alpha\beta}\frac{dx \wedge dy}{xy} \tag{4.37}$$

$\mu_{\alpha\beta}$ must be regular in $U_\alpha \cap U_\beta$. By this condition, the variables other than $c_{13}$ or $c_{24}$ are zero. This can be shown as follows. We denote the general form of the generator as follows

$$x^n y^m dx \wedge dy, \ (n, m \in \mathbb{Z}), \tag{4.38}$$

where $x$ and $y$ are the affine coordinates of $U_1$. For the 2-form (4.38) to be regular in the patch $U_1$, the integers $n$ and $m$ must satisfy

$$n \geq 0, \quad m \geq 0. \quad (U_1) \tag{4.39}$$

In order to determing the regularity condition for $U_2$, we first rewrite the 2-form (4.38) with the affine coordinates $(x', y')$ in the patch $U_2$. The coordinate transformation between $U_1$ and $U_2$ is given in the Figure 4.2.

$$x' = x^{-1}, \quad y' = x^{-1}y \tag{4.40}$$

By this coordinate change, the 2-form (4.38) is transformed as

$$- x'^{-n-m-3} y'^m dx' \wedge dy'. \tag{4.41}$$

The regularity in $U_2$ require that the powers of $x'$ and $y'$ are non-negative, and we obtain the condition

$$m \geq 0, \quad -n-m-3 \geq 0 \quad (U_2) \tag{4.42}$$

In the same way, the condition for $U_3$ and $U_4$ are obtained as follows

$$-m-2 \geq 0, \quad -n-m-3 \geq 0 \quad (U_3), \tag{4.43}$$
$$n \geq 0, \quad -m-2 \geq 0 \quad (U_4). \tag{4.44}$$

These conditions define the allowed region in the lattice $(n,\ m)$ for each patch. The allowed region for $U_i \cap U_j$ can be obtained by the convex hull of two regions for $U_i$ and $U_j$. We can easily see that the allowed regions for $U_1 \cap U_2, U_1 \cap U_4, U_2 \cap U_3, U_3 \cap U_4$ do not contain $(n, m) = (-1, -1)$, and the corresponding coefficients $c_{\alpha\beta}$ are zero.

The general form of $C^1$ is

$$c^1 \ := \ c_{13}\frac{dx \wedge dy}{xy}|_{U_1 \cap U_3} \oplus c_{24}\frac{dx \wedge dy}{xy}|_{U_2 \cap U_4}. \tag{4.45}$$

The condition for anomaly cancellation $\psi_{\alpha\beta\gamma} = 0$ is rewritten as

$$c^2 = \delta c^1, \tag{4.46}$$

where $\delta$ is the coboundary operator. This is equivalent to the following relations among coefficients

$$\begin{aligned} c_{123} &= c_{23} - c_{13} + c_{12}, && (4.47)\\ c_{124} &= c_{24} - c_{14} + c_{12}, && (4.48)\\ c_{134} &= c_{13} - c_{14} + c_{14}, && (4.49)\\ c_{234} &= c_{23} - c_{24} + c_{34}, && (4.50) \end{aligned}$$

where

$$c_{12} = c_{14} = c_{23} = c_{34} = 0. \tag{4.51}$$

Because these equations have the solution

$$c_{13} = -2, \quad c_{24} = 2, \tag{4.52}$$

we conclude that

$$c^2 \in Im(\delta : C^1 \to C^2). \tag{4.53}$$

Therefore the Pontjagin anomaly vanishes in this case.

## 4.3 Riemann-Roch theorem for the 2nd Chern class of del Pezzo surfaces

From the Hirzebruch Riemann-Roch theorem for surface (see Hartshorne Remark 1.6.1 of Chapter V)

$$12(1 + p_a) \;=\; K_X^2 + c_2 \tag{4.54}$$

where $p_a$ is the arithemetic genus of the surface X, which is known to be equal to 0 (see the Theorem 6.2 of Hartshorne Chapter V by Castelnuovo) for rational surfaces. As the del Pezzo surfaces are the special cases of rational surfaces, the LHS = 12. On the other hand, the anticanonical divisor of X is

$$\begin{aligned} -K_X &= 3H - E_1 - \cdots - E_n \\ H^2 &= 1 && (4.55)\\ E_i.E_j &= -\delta_{ij} && (4.56)\\ H.E_i &= 0, && (4.57) \end{aligned}$$

where H is the hyperplane class and $E_i (i = 1, \cdots, n)$ is the exceptional divisor of $0 \le n \le 8$ generic point(s) blowups of $\mathbb{CP}^2$. Therefore we get $K_X^2 = 9 - n$, and the 2nd Chern class $c_2$ is

$$c_2 \quad = \quad 12 - K_X^2 = 3 + n. \tag{4.58}$$

Now that the 2nd Chern character is, keeping in mind that $c_1(X) = -K_X$,

$$\begin{aligned} ch_2(X) &= -c_1(X)^2 + 2c_2(X) \\ &= -(9 - n) + 2(3 + n) \\ &= -3 + 3n. \end{aligned} \tag{4.59}$$

We conclude that the 2nd Chern character vanishes if and only if $n = 1$. We expect that the anomaly 2-form for 1 point blowup should be canceled by the $\mu$-term.

## 4.4 Anomaly 2-forms for 2 points blowups of $\mathbb{CP}^2$

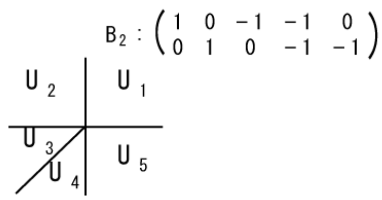


Figure 4.3: Toric diagram for 2 points blowups of $\mathbb{CP}^2$

For $U_1 \to U_2 \to U_3 \to U_1$,

$$\gamma_1^{[1]} \quad = \quad x = (\gamma_1^{[3]})^{-1}(\gamma_2^{[3]})^{-1} \tag{4.60}$$
$$\gamma_2^{[1]} \quad = \quad y = (\gamma_2^{[3]})^{-1} \tag{4.61}$$
$$\gamma_1^{[2]} \quad = \quad x^{-1} \tag{4.62}$$
$$\gamma_2^{[2]} \quad = \quad y \tag{4.63}$$
$$\gamma_1^{[3]} \quad = \quad x^{-1}y = \gamma_1^{[2]}\gamma_2^{[2]} \tag{4.64}$$
$$\gamma_2^{[3]} \quad = \quad y^{-1} = (\gamma_2^{[2]})^{-1} \tag{4.65}$$

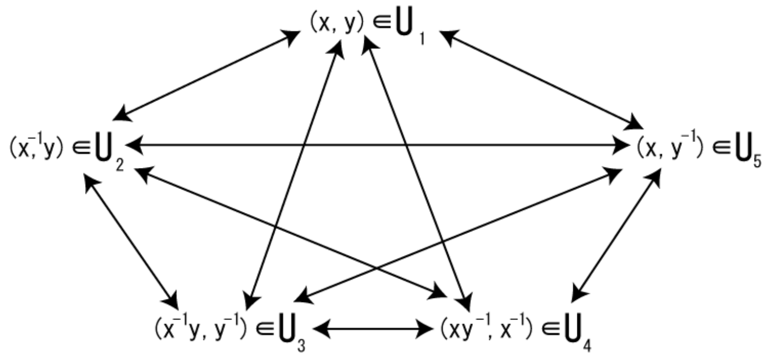


Figure 4.4: Coordinate changes for the 2 points blowups of $\mathbb{CP}^2$

The Jacobians are

$$g = \begin{pmatrix} -x^{-2} & 0 \\ 0 & 1 \end{pmatrix} \tag{4.66}$$

$$\begin{aligned} g' &= \begin{pmatrix} \gamma_2^{[2]} & \gamma_1^{[2]} \\ 0 & -(\gamma_2^{[2]})^{-2} \end{pmatrix} \\ &= \begin{pmatrix} y & x^{-1} \\ 0 & -y^{-2} \end{pmatrix} \end{aligned} \tag{4.67}$$

$$\begin{aligned} g'' &= \begin{pmatrix} -(\gamma_1^{[3]})^{-2}(\gamma_2^{[3]})^{-1} & -(\gamma_1^{[3]})^{-1}(\gamma_2^{[[3]})^{-2} \\ 0 & -(\gamma_2^{[3]})^{-2} \end{pmatrix} \\ &= \begin{pmatrix} -x^2y^{-1} & -x^{-1}y \\ 0 & -y^2 \end{pmatrix} \end{aligned} \tag{4.68}$$

$$\tag{4.69}$$

Then

$$dg = dx \begin{pmatrix} 2x^{-3} & 0 \\ 0 & 0 \end{pmatrix} \tag{4.70}$$

$$dg' = dy \begin{pmatrix} 1 & 0 \\ 0 & 2y^{-3} \end{pmatrix} + dx \begin{pmatrix} 0 & -x^{-2} \\ 0 & 0 \end{pmatrix} \tag{4.71}$$

Therefore

$$dg \wedge dg' = dx \wedge dy \begin{pmatrix} 2x^{-3} & 0 \\ 0 & 0 \end{pmatrix} \tag{4.72}$$

$$\psi_{123}^0 = trg''dg \wedge dg' = -2dx \wedge dyx^{-1}y^{-1} \tag{4.73}$$

$$c_{123} = -2 \tag{4.74}$$

For $U_1 \to U_2 \to U_4 \to U_1$, the affine coordinates are

$$\begin{aligned}
\gamma_1^{[1]} &= x = (\gamma_2^{[4]})^{-1} && (4.75)\\
\gamma_2^{[2]} &= y = (\gamma_1^{[4]})^{-1}(\gamma_2^{[4]})^{-1} && (4.76)\\
\gamma_1^{[2]} &= x^{-1} && (4.77)\\
\gamma_2^{[2]} &= y && (4.78)\\
\gamma_1^{[4]} &= xy^{-1} = (\gamma_1^{[2]})^{-1}(\gamma_2^{[2]})^{-1} && (4.79)\\
\gamma_2^{[4]} &= x^{-1} = \gamma_1^{[2]} && (4.80)
\end{aligned}$$

The Jacobians are

$$\begin{aligned}
(h_i^a)_{ia} &= \begin{pmatrix} -x^{-2} & 0 \\ 0 & 1 \end{pmatrix} && (4.81)\\
h' &= \begin{pmatrix} -(\gamma_1^{[2]})^{-2}(\gamma_2^{[2]})^{-1} & 1 \\ -(\gamma_1^{[2]})^{-1}(\gamma_2^{[2]})^{-2} & \\ 0 & \end{pmatrix} && (4.82)\\
&= \begin{pmatrix} -x^2y^{-1} & 1 \\ -xy^{-2} & \\ 0 & \end{pmatrix} && (4.83)\\
h'' &= \begin{pmatrix} 0 & -(\gamma_1^{[4]})^{-2}(\gamma_2^{[4]})^{-1} \\ -(\gamma_2^{[4]})^{-2} & -(\gamma_1^{[4]})^{-1}(\gamma_2^{[4]})^{-2} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -x^{-1}y^2 \\ -x^2 & -xy \end{pmatrix} && (4.84)
\end{aligned}$$

Then

$$\begin{aligned}
dh &= dx \begin{pmatrix} 2x^{-3} & 0 \\ 0 & 0 \end{pmatrix} && (4.85)\\
dh' &= dy \begin{pmatrix} x^2y^{-2} & 0 \\ 2xy^{-3} & 0 \end{pmatrix} + dx \begin{pmatrix} -2xy^{-1} & 0 \\ -y^{-2} & 0 \end{pmatrix} && (4.86)
\end{aligned}$$

Therefore

$$\begin{aligned}
dh \wedge dh' &= \begin{pmatrix} 2x^{-1}y^{-2} & 0 \\ 0 & 0 \end{pmatrix} dx \wedge dy && (4.87)\\
trh'' dh \wedge dh' &= 0 && (4.88)\\
c_{124} &= 0 && (4.89)
\end{aligned}$$

In the same way, we obtain

$$c_{134} = 1,$$

$$
\begin{aligned}
c_{234} &= -1, \\
c_{125} &= 0, \\
c_{135} &= 0, \\
c_{145} &= -2, \\
c_{235} &= -2, \\
c_{245} &= -2, \\
c_{345} &= -1.
\end{aligned} \tag{4.90}
$$

For $\mu_{\alpha\beta} = c_{\alpha\beta} dx \wedge dy/xy$ to be regular in the intersections $U_i \cap U_j$, only the coefficients $c_{13}, c_{14}, c_{24}, c_{25}$, and $c_{35}$ can be non-vanishing. In this case, $\psi^0_{\alpha\beta\gamma}$ cannot be cancelled by the $\mu$-terms. Namely, there is no solution to the equation

$$
c_{\alpha\beta\gamma} = c_{\beta\gamma} - c_{\alpha\gamma} + c_{\alpha\beta}. \tag{4.91}
$$

The absence of the solution can be checked by making the total anomaly, which is the only gauge invariant linear combination of the 2 cocycles $c_{\alpha\beta\gamma}$. In this case, the total anomaly is $c_{123} + c_{134} + c_{145} = -2 + 1 - 2 = -3$. If we substitute (4.91) into this total anomaly, all the non-vanishing $c_{\alpha\beta}$ cancel, and there is no solution to $c^2 = \delta c^1$.

In general, the total anomaly is given as the sum of $c_{\alpha\beta\gamma}$ for each triangle in a triangulation of the coordinate change diagram (Figure 4.4).

## 4.5 Anomaly 2-forms for 3 points blowups of $\mathbb{CP}^2$

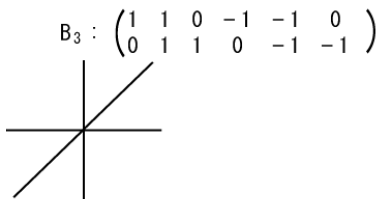


Figure 4.5: Toric diagram for 3 points blowups of $\mathbb{CP}^2$

In this section, we discuss the anomaly for the generic 3 points blowup of $\mathbb{CP}^2$. In the same way in the previous sections, we obtain the following coefficients $c_{\alpha\beta\gamma}$ of $\psi^0_{\alpha\beta\gamma}$.

$$
c_{123} = -1, \quad c_{124} = -2, \quad c_{125} = -2, \quad c_{126} = -1,
$$

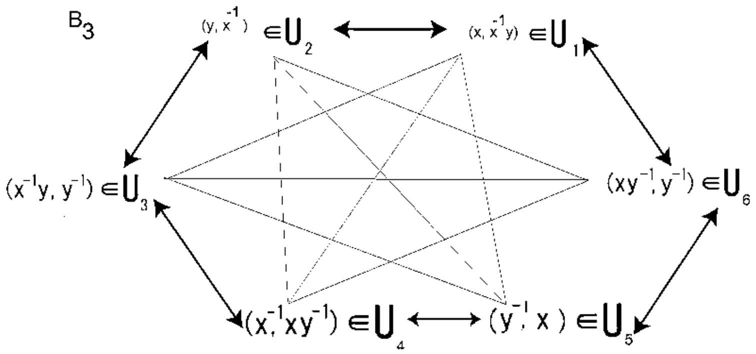


Figure 4.6: Coordinate changes for the 3 points blowups of $\mathbb{CP}^2$

$$\begin{aligned}
&c_{134} = -2, \quad c_{135} = -3, \quad c_{136} = -2, \quad c_{145} = -2, \\
&c_{146} = -2, \quad c_{156} = -1, \quad c_{234} = -1, \quad c_{235} = -2, \\
&c_{236} = -2, \quad c_{245} = -2, \quad c_{246} = -3, \quad c_{256} = -2, \\
&c_{345} = -1, \quad c_{346} = -2, \quad c_{356} = -2, \quad c_{456} = -1.
\end{aligned} \tag{4.92}$$

For the intersections $U_i \cap U_j$, only the coefficients $c_{13}, c_{14}, c_{15}, c_{24}, c_{25}, c_{26}, c_{35}, c_{36}$, and $c_{46}$ are non-vanishing. The total anomaly, the gauge invariant linear combination of $c_{\alpha\beta\gamma}$, is

$$c_{123} + c_{134} + c_{145} + c_{156} = -6. \tag{4.93}$$

From the results of section 3.1, 4.2, 4.4, 4.5, we find that the total anomaly with the proportional constant $-1/2$ is

$$\frac{3}{2}(n-1), \tag{4.94}$$

for $n = 0, 1, 2, 3$ generic points blowup of $\mathbb{CP}^2$ (toric rational del Pezzo surfaces). The 2nd Chern character is in agreement with this discussion.

# Chapter 5

# Conclusion and future direction

In this paper, we explicitly examined by 2 ways that the ansatzes of Witten's heterotic (0,2) model and the Nekrasov's generalized complex geometry are consistent. One way is by step by step careful OPE calculation and the other is the computation of the anomaly 2-form – the 2-cocycle of the chiral de Rham complex – in terms of coordinate transformation Jacobian matrices. We compute the anomaly 2-forms in the case of toric del Pezzo surfaces of all degrees and conclude that this coincides (up to proportional constant 2) with the results of Hirzebruch Riemann-Roch theorem. The most notable case is the 1 point blowup, where the total gauge invariant anomaly vanishes.

Note that Beilinson-Drinfeld chiral algebra has a background of geometric Langlands programs studied on the geometric quantization of vector bundles (stable Higgs bundles), which is similar to physicists' theory of topological fields or topological strings, and Drinfeld's algebro-geometrical construction of Wess-Zumino-Witten model as quantization of 2-dimensional Yang-Mills theory. The author has also worked in this area for several years and in the near future he would like to summarize this result on the automorphic representation theory with physics interpretation in the spirit of Drinfeld, Kapranov-Vasserot's motivic integration theory. This is a bridge between the twist methods of 4 dimensional Super-Yang-Mills theory of Gukov, Kapustin-Witten et.al. and the local geometric Langlands program of Edward Frenkel, Gaitsgory et.al.

During the preparation of this paper, Kapustin-Witten[KW] submitted a paper to the preprint, which is related to this paper.

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# Appendix A

# Wess-Zumino-Witten term

The Wess-Zumino-Witten term is used as a term corresponding to the 1-form $B$, which is used in the Nekrasov's ansatz (2.60). We will make a historical note on this term. First, We will think of the $SU(3)_L \times SU(3)_R$ spontaneously broken to the diagonal $SU(3)$, which involves the Goldstone boson $\pi$.

$$\mathcal{L} = \frac{1}{16\pi} F_\pi^2 \int d^4 x Tr \partial_\mu U \partial_\mu U^{-1}. \tag{A.1}$$

The Euler-Lagrange equation is

$$\partial_\mu \left( \frac{1}{8} F_\pi^2 U^{-1} \partial_\mu U \right) = 0. \tag{A.2}$$

We will add a term, which violate $P_0$ that changes $x \to -x$, $t \to t$, $U \to U$.

$$\partial_\mu \left( \frac{1}{8} F_\pi^2 U^{-1} \partial_\mu U \right) + \lambda \epsilon^{\mu\nu\alpha\beta} U^{-1} (\partial_\mu U) U^{-1} (\partial_\nu U) U^{-1} (\partial_\alpha U) U^{-1} (\partial_\beta U) = 0.$$

We would like to derive this equation from a lagrangian, which is difficult in the first sight.

## A.1 Analogy with particle of magnetic monopoles

The equation of motion in the constrained system $\Sigma x_i^2 = 1$ can have both $x \to -x$ and $t \to -t$ symmetry, if we write it as follows

$$m \frac{\partial^2 x_i}{\partial t^2} + m x_i \left( \sum_k \left( \frac{\partial x_k}{dt} \right)^2 \right) = \alpha \epsilon_{ijk} x_j \frac{\partial x_k}{\partial t}. \tag{A.3}$$

As the right hand side can be seen as the Lorenz force for an electric charge interacting with a magnetic monopole located at the center of the sphere. In

this case the vector potential is, by definition,

$$\nabla \times \vec{A} \;=\; \frac{\vec{x}}{|x|^3}. \tag{A.4}$$

Then by the Stokes Theorem, we can rewrite the contour integral of vector potential as the integration of field strength (flux) trough a topological disk $D$, whose choise can be arbitrary and we can especially take as $D'$ (orientation reversed).

$$1 \;=\; \exp(\sqrt{-1}\alpha \int_{D+D'} F_{ij} d\Sigma^{ij}), \tag{A.5}$$

therefore $\alpha$ is integer or half-integer, which is the Dirac quantization.

## A.2 Return to the original equation of motion of WZW theory

As is noted above, we need some term proportional to the field strength, so we let $Q$ be five-dimensional disc which has the $SU(3)$ as the boundary because of $\pi_4(SU(3)) = 0$

$$\Gamma \;=\; \int_Q \omega_{ijklm} d\Sigma^{ijklm}. \tag{A.6}$$

$Q$ can be deformed to orientation revered $Q'$, then $Q + Q' = S$ is a five-dimensional sphere.

$$\int_S \omega_{ijklm} d\Sigma^{ijklm} \;=\; 2\pi \cdot integer. \tag{A.7}$$

Then $S$ is in $SU(3)$ and $\pi_5(SU(3)) = \mathbb{Z}$ therefore, we can take a unit sphere $S_0$ and write the action $I$ as

$$I \;=\; \frac{1}{16\pi} F_\pi^2 \int d^4x Tr \partial_\mu U \partial_\mu U^{-1} + n\Gamma. \tag{A.8}$$

## A.3 Relation to the Nekrasov ansatz

In the ansatz of Nekrasov(2.60) in section 2.7, we derived the symmetric part $\sigma$ and antisymmetirc part $\mu$ of the 1-form $B$. If we combine them, we obtain

$$\tilde{\beta}_a = \beta_i g^i_a + \frac{1}{2} tr(\mathcal{G}_a g \partial g^{-1}) + \iota_{\tilde{\partial}_a} \mu, \tag{A.9}$$

where $\iota$ is the inner derivative. If we rewrite (A.9) by multiplying $\bar{\partial}\tilde{\gamma}^a/2\pi$ from the right and use the cyclicity of trace, we obtain

$$\frac{1}{2\pi}\tilde{\beta}\bar{\partial}\tilde{\gamma} = \frac{1}{2\pi}\beta\bar{\partial}\gamma + L_{WZW}(g), \tag{A.10}$$

where $L_{WZW}(g)$ is the level one Wess-Zumino-Witten Lagrangian

$$L_{WZW}(g) = \frac{1}{4\pi} tr(g^{-1}\partial g g^{-1}\bar{\partial} g) + \frac{1}{12\pi} d^{-1} tr[(g^{-1}dg)^3]. \tag{A.11}$$

# Appendix B

# Toric diagrams and birational geometry

Toric varieties are algebraic manifolds which have the action of algebraic torus $(\mathbb{C}^\times)^n = (\mathbb{C} \setminus \{0\})^n$, where n is the complex dimension of the varieties. For each toric diagram (fan), we can define the following data. Vertices $e_i \in \mathbb{Z}^2$, and vectors $v_i$ from the origin to the vertices. The dual basis $w_i^1, w_i^2$ for affine patch $U_i$ spaned by the 2 vectors $v_i, v_{i+1}$ is such that

$$(v_i, w_i^1) = 1, \quad (v_i, w_i^2) = 0, \quad (v_{i+1}, w_i^1) = 0, \quad (v_{i+1}, w_i^2) = 1. \tag{B.1}$$

The affine patch $U_i$ is described by

$$Spec\mathbb{C}[x^{(w_i^1,E_1)}y^{(w_i^1,E_2)}, x^{(w_i^2,E_1)}y^{(w_i^2,E_2)}], \tag{B.2}$$

where $E_1 = (1,0)$, and $E_2 = (0,1)$ are the standard bases of $\mathbb{Z}^2$. Namely the canonical coordinates $v^{[i]}, w^{[i]}$ in the affine patch $U_i$ are given by

$$v^{[i]} = x^{(w_i^1,E_1)}y^{(w_i^1,E_2)} \tag{B.3}$$

$$w^{[i]} = x^{(w_i^2,E_1)}y^{(w_i^2,E_2)}. \tag{B.4}$$

The blowups are operations in the divisor linear system such that we assign a new vertex $e_{N+1} \in \mathbb{Z}^2$ ($N \in \mathbb{Z}$ is the number of original vertices) in the generic point and the corresponding vector from the origin. This operation is birational since we have the toric action on the toric variety, which includes the toric variety $(\mathbb{C}^\times)^2$ as the dense submanifold and therefore the toric diagram in $\mathbb{Z}^2$ has the $GL(2)$ action.

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